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Solutions of Higher Order Boundary Value Problems by Homotopy Perturbation Method

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ABSTRACT

In this paper, homotopy perturbation method is applied to the high-order nonlinear boundary value problems namely fourth-order and seventh-order. The numerical results are compared to the exact solution to verify the accuracy of the method used. The results obtained with minimum amount of computational work show that the present method is efficient and convenient.

Keywords: Homotopy Perturbation Method; Nonlinear Boundary Value Problems; Exact solutions.

1. Introduction

Until now, nonlinear analytical methods for solving nonlinear problems have been conquered by perturbation methods. Perturbation methods are based on assumptions that need the existence of a small parameter in an equation Nayfeh (2000). This small parameter will give an ideal result if it is suitable and vice versa. Thus it is essential to develop new nonlinear method which does not require small parameters at all. A great attention has been given towards the application of homotopy perturbation method in nonlinear problems. This is related to the fact that homotopy perturbation scheme is continuously deform the nonlinear into easier and solvable linear form of equation. HPM is a method developed by merging the homotopy with perturbation technique. It has been proposed by He with the aim to solve linear and nonlinear, initial and boundary value problems. The method has many advantages such as the solution is given in an infnite series that rapidly converge to the exact solution, deforms the difficult problem into simpler one for easy computational work, less time taken to get the solutions, and the method is very straightforward as the procedure can be done using pencil and paper only He (2006). HPM has solved many linear and nonlinear problems including integral equations Abbasbandy (2006b), quadratic Riccati differential equation Abbasbandy (2006a), stiff system of ordinary differential equation Aminikhah (2011), partial differential algebraic equations Jafari (2010) and many more. Boundary value problems (BVPs) appear in many fields such as physics, engineering, chemistry and medicine. It has been an active research undertaking in finding the solution of high-order nonlinear boundary value problems using various numerical methods due to complexity and strong nonlinearity. Adomian decomposition method (ADM) has solved third, fourth and fifth BVPs Hasan (2012), Hashim (2006), Wazwaz (2001), Shahid et.al. use differential transformation method (DTM)to find the solution of seventh order BVPs Siddiqi (2012). Also, the solution of eighth-order BVPs are provided by Shahid and Twizell using polynomial spline Siddiqi (1996) as well as using finite difference methods developed by Boutayeb and Twizell Boutayeb (1993). Recently, Chowdhury *et al.* Chowdhury (2009a), Chowdhury and Hashim Chowdhury (2009b) were the first to successfully apply the multistage homotopy-perturbation method (MHPM) to the chaotic Lorenz system and Chen system. The multistage HPM (MPHM) Chowdhury (2012a,b), Chowdhury et al. (2015), Hashim (2008a,b) is a powerful technique to get more reliable and efficient approximate solutions for chaotic and hyperchaotic problems. It is an improvement over the standard HPM.Very recently, Chowdhury et al Chowdhury (2013) introduced modified HPM to solve differential and integral equations. In this work, we are interested to apply the new technique of choosing initial approximation to the BVPs.

2. Solution Approach

According to Siddiqi (2014), consider the following n-th order boundary value problem

$$y^{(n)}(x) = f(x, y, y', \cdots, y^{(n-1)}), \tag{1}$$

with boundary conditions

$$y(a) = \alpha_0, y'(a) = \alpha_1, y''(a) = \alpha_2, \cdots, y^{(n-1)}(a) = \alpha_{n-1}, \tag{2}$$

$$y(b) = \beta_0, y'(b) = \beta_1, \tag{3}$$

where f is a continuous function on $[a, b] \times D$, D is an open subset of R^{n-1}, while the parameters α_i, β_0 and β_1 are real numbers. Construct the homotopy as follows

$$u^{(n)}(x) = pf(x, u, u', \cdots, u^{(n-1)}). \tag{4}$$

Here $p \in [0, 1]$ is the embedding parameter. The assumption of solution for Eq. (1)

$$u = u_0 + pu_1 + p^2 u_2 + \cdots, \tag{5}$$

If the nonlinear term exists in the equation, it can be expressed as

$$N(u) = N(u_0) + pN(u_0, u_1) + p^2 N(u_0, u_1, u_2) + \cdots, \tag{6}$$

where

$$N(u_0, u_1, \cdots, u_n) = \frac{1}{n!}\frac{d^n}{dp^n}[N(\sum_{k=0}^{n} p^k u_k)]_{p=0}, n = 0, 1, 2, \cdots. \tag{7}$$

Equation Eq. (7) is called He's polynomial . Substitute Eq. (4) into Eq. (5) and arrange the coefficient of same power of p

$$u_0^{(n)} = 0, u_0^{(i)}(a) = A, u_0^{(n-2)}(a) = B, u_0^{(n-1)}(a) = C, i = 0 \cdots (n-1), \tag{8}$$

$$u_1^{(n)} = f(x, u_0, u_0', \cdots, u_0^{(n-1)}), u_0^{(i)}(a) = 0, i = 0 \cdots (n-1), \tag{9}$$

$$u_2^{(n)} = f(x, u_0, u_1, u_0', u_1', \cdots, u_0^{(n-1)}, u_1^{(n-1)}), u_1^{(i)}(a) = 0, i = 0 \cdots (n-1), \tag{10}$$

etc where A, B, C and so on are the initial conditions to be determined. We substitute with the unknowns when the initial conditions are not given. The approximations of solutions for Eq. (8) - Eq. (10) are

$$u_0 = \sum_{k=0}^{n-1} \frac{u^{(k)}(a)}{k!} x^k, \tag{11}$$

$$u_1 = \int_a^x \int_a^{x_1} \cdots \int_a^{x_{(n-1)}} f(\tau, u_0, u_0', \cdots, u_0^{(n-1)}) dx_{(n-1)} \cdots dx_1 d\tau, \tag{12}$$

$$u_2 = \int_a^x \int_a^{x_1} \cdots \int_a^{x_{(n-1)}} f(x, u_0, u_1, u_0', u_1', \cdots u_0^{(n-1)}) dx_{(n-1)} \cdots dx_1 d\tau, \tag{13}$$

etc. Thus, the solutions can be written as follows

$$v \simeq u_0 + u_1 + u_2 + \cdots . \tag{14}$$

By imposing the boundary conditions given in the problem to the (Eq. (14)) the values of A, B, C and so on can be determined. Then insert the values of A, B, C and so on into the final solutions in (Eq. (14)). Note that the values of A, B, C and so on depend on how many terms we use in the series solutions.

3. Result and Discussion

3.1 Fourth-Order Differential Equation with Product Nonlinearity

Consider the four-point BVP for the fourth-order nonlinear differential equation with product nonlinearity (Duan, 2011),

$$u^4(x) + u(x)u'(x) - 4x^7 - 24 = 0, \quad 0 \leq x \leq 1, \tag{15}$$

with the boundary conditions

$$u(0) = 0, \quad u'(0.25) = 6, \quad u''(0.5) = 3, \quad u(1) = 1. \tag{16}$$

The exact solution is

$$u(x) = x^4. \tag{17}$$

The homotopy is constructed as follows

$$u^4(x) = 4x^7 + 24 - p(uu'(x)). \tag{18}$$

Here $p \in [0,1]$ is the embedding parameter. The assumption of solution

$$u = u_0 + pu_1 + p^2u_2 + \cdots . \tag{19}$$

The nonlinear term uu' is solved using He's polynomial. Substitute Eq. (19) into Eq. (18) and arrange the coefficient of same power of p. For 3 terms, we get

$$u_0^4 = 4x^7 + 24, u_0(0) = 0, u_0'(0) = A, u_0''(0) = B, u_0'''(0) = C, \tag{20}$$

$$u_1^4 = -u_0u_{0_x}, u_1(0) = 0, u_1'(0) = 0, u_1''(0) = 0, u_1'''(0) = 0, \tag{21}$$

$$u_2^4 = u_0u_{1_x} - u_1u_{0_x}, u_2(0) = 0, u_2'(0) = 0, u_2''(0) = 0, u_2'''(0) = 0, \tag{22}$$

etc where A, B and C are unknowns to be determined. The solutions of equations Eq. (20)-Eq. (22) are

$$u_0 = \frac{1}{1980}x^{11} + x^4 + \frac{1}{6}Cx^3 + \frac{1}{2}Bx^2 + Ax, \tag{23}$$

In this example, we provide solution of 1-term, 2-terms and 3-terms are respectively,

$$y_0 = u_0(x), \tag{24}$$

$$y_1 = u_0(x) + u_1(x), \tag{25}$$

$$y_2 = u_0(x) + u_1(x) + u_2(x). \tag{26}$$

Impose the boundary conditions given in the problem to Eq. (24)-Eq. (26) respectively, the values of A, B and C are obtained for each 1-term, 2-term and 3-term are,

$$A = -451.43 \times 10^{-6}, B = -104.69 \times 10^{-6}, C = -7.63 \times 10^{-6}, \tag{27}$$

$$A = 103.11 \times 10^{-9}, B = 699.93 \times 10^{-12}, C = 101.25 \times 10^{-12}, \tag{28}$$

$$A = -16.75 \times 10^{-12}, B = -36.65 \times 10^{-15}, C = -16.36 \times 10^{-15}. \tag{29}$$

Substitute the Eq. (27) into Eq. (24), we get the solution of 1-term HPM.

$$\begin{aligned} y_0 = {} & 510.00 \times 10^{-6}x^{11} + x^4 - 1.27 \times 10^{-6}x^3 \\ & -52.34 \times 10^{-6}x^2 - 451.43 \times 10^{-6}x. \end{aligned} \tag{30}$$

Substitute the Eq. (28) into $u0$ Eq. (24) and $u1$ Eq. (25),

$$u_0 = 510.00 \times 10^{-6}x^{11} + x^4 + 16.88 \times 10^{-12}x^3 +349.96 \times 10^{-12}x^2 + 103.11 \times 10^{-9}x, \quad (31)$$

$$u_1 = -9.24 \times 10^{-12}x^{25} - 103.20 \times 10^{-9}x^{18} - 2.09 \times 10^{-18}x^{17} -52.60 \times 10^{-18}x^{16} - 19.07 \times 10^{-15}x^{15} - 510.00 \times 10^{-6}x^{11} -23.44 \times 10^{-15}x^{10} - 694.38 \times 10^{-15}x^9 - 306.88 \times 10^{-12}x^8 -8.58 \times 10^{-21}x^7 - 300.71 \times 10^{-21}x^6 - 88.60 \times 10^{-18}x^5, \quad (32)$$

and substitute Eq. (31)-Eq. (32) into Eq. (25) the series solution for 2-term become,

$$y_1 = x^4 + 1.6875000000 \times 10^{-13}x^3 + 3.4996500000 \times 10^{-10}x^2 +1.0311000000 \times 10^{-7}x - 9.2418845164 \times 10^{-12}x^{25} -1.0315574041 \times 10^{-7}x^{18} - 2.0889037430 \times 10^{-18}x^{17} -5.2604166670 \times 10^{-17}x^{16} - 1.9075369080 \times 10^{-14}x^{15} -2.3437500000 \times 10^{-14}x^{10} - 6.9437500000 \times 10^{-13}x^9 -3.0687500000 \times 10^{-10}x^8 - 8.5772333360 \times 10^{-21}x^7 -3.0070742620 \times 10^{-19}x^6 - 8.8597267500 \times 10^{-17}x^5. \quad (33)$$

Note that the values of parameter A, B and C depend on the term used in the calculations. In table 1 to 3 the error between solution of 1-term, 2-term and 3-term HPM are compared to the exact solution respectively. The error is reduced as the term increased. The graphs of error are plotted in figure 4 for 1-term, figure 5 for 2-term and figure 6 for 3-term solution. When 3-term is used the solution of HPM become equal to the exact solution at $x = 0.6$ up till $x = 1$ showing the rapid convergence of HPM. Then, the solution for 3-term HPM and exact solution is plotted in figure 7 showing that the solution of HPM converges to the exact solution for $0 \leq x \leq 1$.

Table 1: The solution of 3-term HPM, exact and the absolute error

x	$u(x)$ *exact*	$u(x)$ *HPM*	*Absolute* Error
0.0	0.0000000000	0.0000000000	0
0.1	0.0001000000	0.00005433227833	4.56677216700E-05
0.2	0.0016000000	0.001507610037	9.23899630000E-05
0.3	0.0081000000	0.00795982651	0.00014017349
0.4	0.0256000000	0.02541099259	0.00018900741
0.5	0.0625000000	0.0622612864	0.0002387136
0.6	0.1296000000	0.1293118554	0.0002881446
0.7	0.2401000000	0.2397679002	0.0003320998
0.8	0.4096000000	0.4092480876	0.0003519124
0.9	0.6561000000	0.6558088768	0.0002911232
1.0	1.0000000000	1.000000004	4.000000E-09

Table 2: The solution of 2-term HPM, exact and the absolute error

x	$u(x)$ *exact*	$u(x)$ *HPM*	*Absolute* Error
0.0	0.0000000000	0.0000000000	0
0.1	0.0001000000	0.0001000103145	1.03145000E-08
0.2	0.0016000000	0.001000206360	2.06360000E-08
0.3	0.0081000000	0.008100030964	3.09640000E-08
0.4	0.0256000000	0.025600041300	4.13000000E-08
0.5	0.0625000000	0.06250006164	5.16400000E-08
0.6	0.1296000000	0.12960006200	6.20000000E-08
0.7	0.2401000000	0.2401000722	7.22000000E-08
0.8	0.4096000000	0.4096000807	8.07000000E-08
0.9	0.6561000000	0.65610007750	7.75000000E-08
1.0	1.0000000000	0.9999999995	5.00000000E-10

Table 3: The solution of 3-term HPM, exact and the absolute error

x	$u(x)$ *exact*	$u(x)$ *HPM*	*Absolute* Error
0.0	0.0000000000	0.0000000000	0
0.1	0.0001000000	0.00009999999832	1.6800E-12
0.2	0.0016000000	0.00159999999700	3.0000E-12
0.3	0.0081000000	0.00809999999500	5.0000E-12
0.4	0.0256000000	0.02559999999000	1.00000E-11
0.5	0.0625000000	0.06249999999000	1.00000E-11
0.6	0.1296000000	0.1296000000	0
0.7	0.2401000000	0.2401000000	0
0.8	0.4096000000	0.4096000000	0
0.9	0.6561000000	0.6561000000	0
1.0	1.0000000000	1.0000000000	0

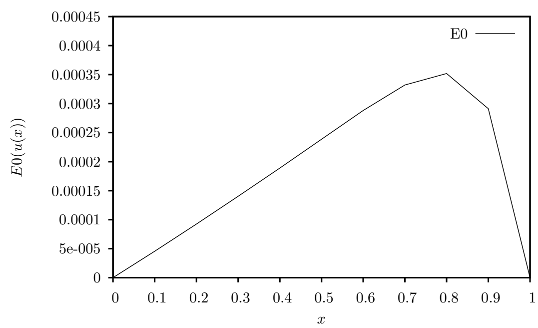

Figure 1: The error between 1-term HPM and exact solution.

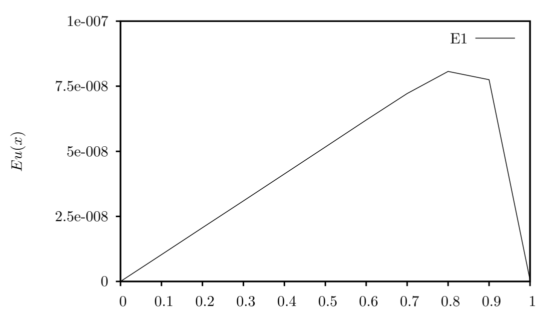

Figure 2: The error between 2-term HPM and exact solution.

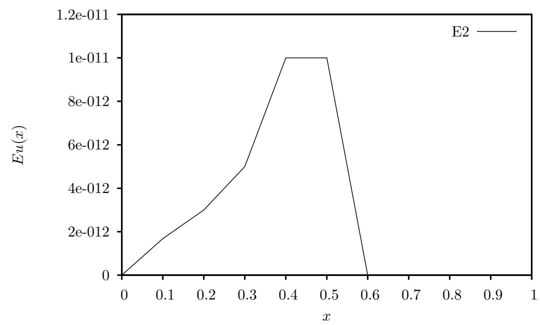

Figure 3: The error between 3-term HPM and exact solution.

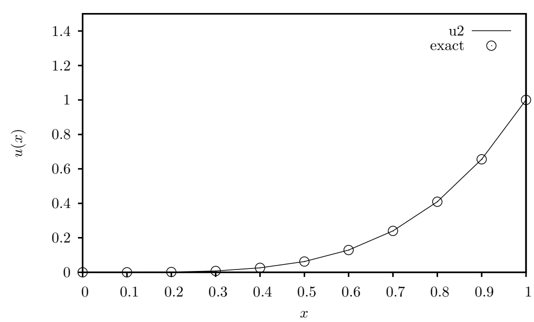

Figure 4: The solution of 3-term HPM and exact solution.

3.2 Seventh Order Nonlinear BVP

Consider the following seventh-order BVP (Inc, 2014),

$$u^7(x) = u(x)u'(x) + e^{-2x}(2 + e^x(x-8) - 3x + x^2), \quad 0 \le x \le 1, \tag{34}$$

with the boundary conditions

$$\begin{aligned} u(0) = 1, \quad u'(0) = -2, \quad u''(0) = 3, \quad u'''(0) = -4, \\ u(1) = 0, \quad u'(1) = -\frac{1}{e}, \quad u''(1) = \frac{2}{e}. \end{aligned} \tag{35}$$

The exact solution is

$$u(x) = (1-x)e^{-x}. \tag{36}$$

The homotopy is constructed as follows

$$u^7(x) - v_0^7(x) + p(v_0^7(x) - u(x)u'(x) - e^{-2x}(2 + e^x(x-8) - 3x + x^2)), \tag{37}$$

initialize the unknown initial conditions as

$$u^4(0) = A, \quad u^5(0) = B, \quad u^6(0) = C. \tag{38}$$

In the earlier works Chowdhury et.al. Chowdhury (2013), introduced an alternative of choosing the initial approximations

$$v_0 = L^{-1}(g(x)) + \phi x = f(x), \tag{39}$$

where the function $f(x)$ represents the terms arising from integrating the source term $g(x)$ and from using the given conditions $\phi(x)$, all of which are assumed to be prescribed. The nonlinear term $Nu_k = F(u)$ is usually represented by an infinite series of the so-called He's polynomials. The modified form is based on the assumption that the initial approximation v_0 given in Eq. (39) can be decomposed into two parts, namely f_0 and f_1 such that $f = f_0 + f_1$. Under this assumption ,

$$u_0(x) = f_0(x), \tag{40}$$

$$p^1 : u_1 = f_1 - L^{-1}(Ru_0) - L^{-1}(Nu_0), \tag{41}$$

$$\begin{aligned} p^{k+2} : u_{k+2}(x) &= -L^{-1}(Ru_{k+1}) - L^{-1}(Nu_{k+1}) \\ &= -L^{-1}(Ru_{k+1}) - L^{-1}(H_{k+1}), k \geq 0. \end{aligned} \tag{42}$$

Using the method proposed,

$$\begin{aligned} v_0 = L^{-1}(e^{-2x}(2 + e^x(x-8) - 3x + x^2)) + 1 - 2x \\ + \frac{3}{2}x^2 - \frac{2}{3}x^3 + \frac{Ax^4}{24} + \frac{Bx^5}{120} + \frac{Cx^6}{720}, \end{aligned} \tag{43}$$

where $L = \frac{\partial^7}{\partial x^7}$ and $L^{-1} = \int_0^x \int_0^x \int_0^x \int_0^x \int_0^x \int_0^x \int_0^x (.) dxdxdxdxdxdxdx$ and

$$v_0 = f_0 + f_1. \tag{44}$$

Following Eq. (40)-Eq. (42) for 3 terms,

$$u_0(x) = f_0(x), \tag{45}$$

$$u_1 = f_1(x) + L^{-1}(Nu_0), \tag{46}$$

$$u_2 = L^{-1}(Nu_1). \tag{47}$$

The nonlinear term $u(x)u'(x)$ is given by the He's polynomial. Thus,

$$u_0 = 1, \tag{48}$$

$$\begin{aligned} u_1 := -\frac{1}{11520}\Big(&11025e^{2x} - 22410xe^{2x} + 16920x^2e^{2x} - 7560x^3e^{2x} \\ &+2370x^4e^{2x} - 564x^5e^{2x} + 104x^6e^{2x} + 495 + 11520xe^{x} \\ &-11520e^{x} + 90x^2 + 360x\Big)e^{-2x} + \frac{1}{720}Cx^6 \\ &+\frac{1}{120}Bx^5 + \frac{1}{24}Ax^4 - \frac{2}{3}x^3 + \frac{3}{2}x^2 - 2x, \end{aligned} \tag{49}$$

$$\begin{aligned} u_2 := \frac{1}{3832012800}\Big(&19173161700e^{2x} - 15349568850xe^{2x} + \\ &5765326875x^2e^{2x} - 1286381250x^3e^{2x} + 163097550x^4e^{2x} - \\ &997920x^5e^{2x} - 5093550x^6e^{2x} - 41580x^7e^{2x} + 5940x^8e^{2x} - \\ &3832012800xe^{x} - 19160064000e^{x} - 4677750x - 13097700 - \\ &467775x^2 - 660x^9e^{2x} - 5214x^{10}e^{2x} + 564x^{11}e^{2x} - 52x^{12}e^{2x} \\ &+8Cx^{12}e^{2x} + 96Bx^{11}e^{2x} + 1056x^{10}e^{2x}\Big)e^{-2x}. \end{aligned} \tag{50}$$

By imposing the boundary conditions given, the values of A, B and C are obtained

$$A = 5.002746746, \quad B = -6.039224163, \quad C = 7.197868078 \tag{51}$$

The solution of 3 terms of HPM is

$$\begin{aligned} y_2 = 1 - \frac{1}{11520}\Big(&11025e^{2x} - 22410xe^{2x} + 16920x^2e^{2x} - 7560x^3e^{2x} \\ &+2370x^4e^{2x} - 564x^5e^{2x} + 104x^6e^{2x} + 495 + 11520xe^{x} - 11520e^{x} \\ &+90x^2 + 360x\Big)e^{-2x} + 0.009997038997x^6 - 0.05032686802x^5 \\ &+0.2084477811x^4 - \frac{2}{3}x^3 + \frac{3}{2}x^2 - 2x + \frac{1}{3832012800} \\ &\Big(19173161700e^{2x} - 15349568850xe^{2x} + 5765326875x^2e^{2x} \\ &-1286381250x^3e^{2x} + 163097550x^4e^{2x} - 997920x^5e^{2x} - 5093550x^6e^{2x} \\ &-41580x^7e^{2x} + 5940x^8e^{2x} - 3832012800xe^{x} - 19160064000e^{x} \\ &-4677750x - 13097700 - 467775x^2 - 660x^9e^{2x} + 68.900564x^{10}e^{2x} \\ &-15.7655196x^{11}e^{2x} + 5.58294462x^{12}e^{2x}\Big)e^{-2x}. \end{aligned} \tag{52}$$

Table 4: The solution of 3-term HPM, exact and the absolute error

x	$u(x)$ *exact*	$u(x)$ *HPM*	*Absolute error*
0.0	1.0000000000	1.0000000000	0
0.1	0.8143536762	0.8143536841	7.9000000E-09
0.2	0.6549846025	0.6549847014	9.8900000E-08
0.3	0.5185727545	0.5185730824	3.2790000E-07
0.4	0.4021920276	0.4021926803	6.5270000E-07
0.5	0.3032653298	0.3032662574	9.2760000E-07
0.6	0.2195246544	0.2195256434	9.8900000E-07
0.7	0.1489755911	0.14897636	7.6890000E-07
0.8	0.08986579282	0.08986618468	3.91860E-07
0.9	0.04065696597	0.04065704395	7.79800E-08
1.0	0	−0.0000000012505	1.25050E-09

The exact solution and solution of HPM together with the absolute error for the first 3 terms are presented in table 8. From the error it shows that the solution of HPM agrees to the exact solution using only 3 terms. In figure 9 the absolute error is plotted from $x = 0$ to $x = 1$ while in figure 10 the exact solution and the solution of HPM are plotted.

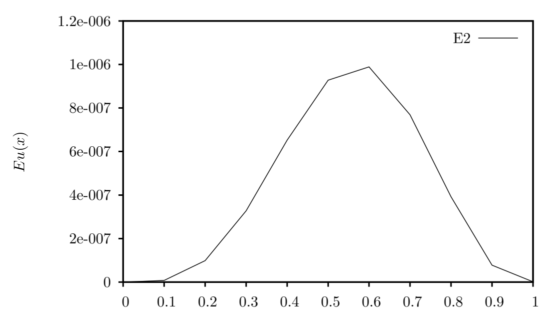

Figure 5: The error between 3-term HPM and exact solution.

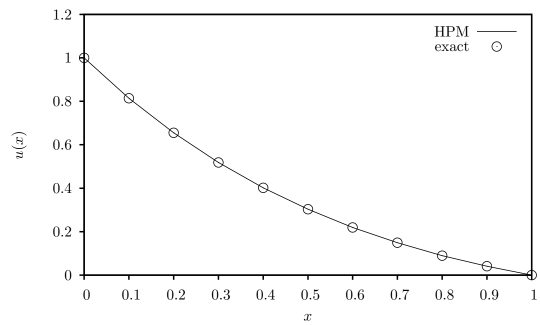

Figure 6: The solution of 3-term HPM and exact solution.

4. Conclusions

In this task, the high-order nonlinear BVPs are solved accurately by HPM using only a few iterations. The technique of deciding the initial approximation is important in HPM.Unlike the traditional methods, the solutions here are given in series form. The approximate solution to the equation was computed without any need for special transformations, linearization or discretization.Comparison with the exact solutions shows that the homotopy-perturbation method is a promising tool for finding approximate analytical solutions to strongly nonlinear BVPs.

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